Algebraic geometry 2

Exercise Sheet 7

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Exercise 1. (1) Let A be a ring. Show that giving a scheme X over A is equivalent to giving a scheme X and the structure of a sheaf of A-algebras on $\mathcal{O}_X$.
(2) Show that every scheme X is in a unique way a $\mathbb{Z}$-scheme.

Exercise 2. (1) Let A be a ring and X a scheme over A. Show that X is locally of finite type over A if and only if for any open affine subscheme $\operatorname{Spec} B \subset X$, B is a finitely generated A-algebra.
Hint: Use Lemma 3.19.
(2) Deduce from (1) that an affine scheme $\operatorname{Spec} B$ is of finite type over A if and only if B is a finitely generated A-algebra.

Exercise 3. Let X be a scheme. We define the *residue field* of a point $x \in X$ to be the field $k(x) := \mathcal{O}_{X,x}/m_x$, where m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.
Let K be a field. Recall that $\operatorname{Spec} K$ consists of one point.
Show that to give a morphism from $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion of fields $k(x) \hookrightarrow K$. That is, there is a bijection

$$\operatorname{Hom}_{sch}(\operatorname{Spec} K, X) \simeq \{(x, \alpha) \mid x \in X, \alpha : k(x) \hookrightarrow K\}.$$

Exercise 4. Let k be a field. Let $\pi : X \to \operatorname{Spec} k$ and $\rho : Y \to \operatorname{Spec} k$ be k-schemes. A morphism of k-schemes $f : X \to Y$ is a morphism of schemes that is compatible with the structural morphism, that is $\rho \circ f = \pi$. Denote

$$X(k) := \operatorname{Hom}_{k-\text{schemes}}(\operatorname{Spec} k, X)$$

(1) Show that $X(k)$ can be identified with the set of points $x \in X$ such that $k(x) = k$.
Remark: With the identification above, the points from $X(k)$ are called *k-rational points* of X.
(2) Show that $U(k) = X(k) \cap U$ for any open subscheme $U \subset X$.
(3) Let $f_1, \dots, f_m \in k[X_1, \dots, X_n]$ and let $X = \operatorname{Spec} k[X_1, \dots, X_n]/(f_1, \dots, f_m)$. Show that the k-rational points of X correspond to the solutions of polynomial equations $f_1(x) = \dots = f_m(x) = 0$ over k. That is, we have a bijection

$$X(k) \simeq \{x \in k^n \mid f_1(x) = \dots = f_m(x) = 0\}.$$

(4) Consider an affine scheme $X = \operatorname{Spec} \mathbb{R}[X, Y]/(X^2 + Y^2 + 1)$ over $\mathbb{R}$.

(a) Show that $X(\mathbb{R}) = \emptyset$.
(b) Find a point $x \in X$, such that there is a morphism $i : \operatorname{Spec} \mathbb{C} \to X$ of schemes and the image of i is $\{x\}$. Is i the only morphism $\operatorname{Spec} \mathbb{C} \to X$ with such image?